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Upper Bound on the Capacity of Constrained Three-Dimensional Codes

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Abstract — An upper bound on the capacity of constrained three-dimensional codes is presented. The bound for two-dimensional codes of Calkin and Wilf was extended to three dimensions by Nagy and Zeger. Both bounds apply to first order symmetric constraints. The bound in three dimensions is generalized in a weaker form to higher order and non-symmetric constraints.

I. INTRODUCTION

In this paper we consider the capacity of constrained three-dimensional (3-D) codes defined by a set of constraints. We consider shift invariant constraints of finite extent (N, M, L) , in the sense that the constraints may be defined on an N by M by L volume. Each element is taken from an alphabet A of size $|A|$. The $|A|^{NML}$ possible configurations on the volume are divided into a set of admissible and a set of non-admissible configurations. Let $F(n, m, l)$ be the number of distinct admissible configurations (or codewords) on an n by m by l volume not violating the constraints within the volume. The per symbol capacity (or maximum entropy), $C^{(3)}$ of the 3-D code defined by the constraints may be defined as:

$$C^{(3)} = \lim_{n, m, l \rightarrow \infty} \frac{\log F(n, m, l)}{nml} \quad (1)$$

A more formal treatment of the entropy definition and its existence is given in [1].

Calkin and Wilf [2] presented a method giving tight bounds on capacity for the (hard square) 2-D constraint, with binary elements, specified by that for any two 4-neighbors, i.e. horizontal and vertical neighbors, both of them can not be '1'. The upper bound [2] is based on

$$\Lambda \leq \text{Trace}(\mathbf{T}^{2p})^{1/2p}, p > 0. \quad (2)$$

where Λ is the largest eigenvalue of \mathbf{T} . (2) is valid for real symmetric matrices and it is applied to the transfer matrix of the constraint in one direction. Nagy and Zeger [3] extended the results to the 3-D version of the constraint above. (Two direct neighboring '1's in the direction of the third axis is also non-admissible.) Let D denote the dimension of the constraint. Their methods may be applied to other constraints, but they are restricted to constraints for which the transfer matrices are symmetric in at least $D - 1$ directions. This is satisfied for constraints which are of 1st order and symmetric in (at least) all but one direction. Here we address the problem of bounding capacity for higher order and non-symmetric constraints in 3-D, eg. limits on run-lengths or distances (≥ 3).

II. UPPER BOUND FOR HIGHER ORDER 3-D CONSTRAINTS

In order to achieve an upper bound we shall specify a source which has the required symmetric transfer matrices and as a subset can generate all configurations admissible by the original constraint. In [4] we presented a way to do this in 2-D. Extending to 3-D leads to the following construction. The states are defined by the admissible configurations within 4 sub-states, which are rectangular boxes of equal size. The sub-states forming one state must have the same boundary configuration of width $M - 1$ in the m -direction and $L - 1$ in the l -direction. The states extend $N - 1$ in the n -direction. The admissible transitions between the combined states in all generating s_1 by s_2 distinct elements are specified by \mathbf{G}_{s_1, s_2} . The transitions are admissible iff the transitions of the 4 sub-states are.

Theorem 1: The capacity of a 3-D code specified by shift invariant constraints of finite extent (N, M, L) , has the upper bound

$$C^{(3)} \leq \frac{H''(s_1, s_2)}{s_1 s_2} \quad (3)$$

where H'' is the capacity determined by the logarithm of the largest eigenvalue of \mathbf{G}_{s_1, s_2} of the given constraint. s_1 and s_2 are positive even integers.

The principles of the proof is as follows. (2) is applied first in one and then in another direction as in [3]. We need to ensure that the matrices are symmetric. Given a non-symmetric transfer matrix \mathbf{T} (in one direction), introduce $\mathbf{A} = \mathbf{T}^p$ and the symmetric matrix $\mathbf{C} = \mathbf{A} + \mathbf{A}^*$, where $*$ denotes the transpose. Applying (2) to \mathbf{C}^2 , the bound is asymptotically dominated by $\text{Trace}(\mathbf{A}\mathbf{A}^*)$. \mathbf{A}^* may be described as the reverse transition of \mathbf{A} . So the trace counts configurations which are given by two transitions starting and ending in the same state. Used in two directions leads to the construction above and \mathbf{G}_{s_1, s_2} .

We expect to achieve improved numerical results using (3) in 3-D as we did in 2-D [4] using the same approach to derive symmetric transfer matrices generating all admissible configurations as a subset.

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